HW #5

1) $\frac{2}{5}+\frac{2}{8}+\frac{2}{11}+\frac{2}{14}+\dots = \sum_{n=1}^{\infty} \frac{2}{3n+2} = 2\sum_{n=1}^{\infty}\frac{1}{3n+2}$

We compute $\int_1^{\infty} \frac{2}{3x+2}\,dx = \left[\frac{2}{3}\ln(3x+2)\right]_1^{\infty} = \lim_{x\to\infty} \frac{2}{3}\ln(3x+2) - \frac{2}{3}\ln(5)$

diverges to ∞ so $\sum_{n=1}^{\infty}\frac{2}{3n+2}$ diverges by the integral test

Alternatively, $\sum_{n=1}^{\infty}\frac{2}{3n+2} = \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n+2/3} \geq \frac{2}{3}\sum_{n=2}^{\infty}\frac{1}{n}$.

diverges by the squeeze theorem.

2) $\sum_{n=1}^{\infty}\frac{n^2}{\sqrt{1+n^4}}$ Note $\lim_{n\to\infty}\frac{n^2}{\sqrt{1+n^4}} = \lim_{n\to\infty}\frac{\sqrt{n^4}}{\sqrt{1+n^4}} = \lim_{n\to\infty}\sqrt{\frac{n^4}{1+n^4}}$

diverges by the divergence test $= 1$

3) $\sum_{n=2}^{\infty}\frac{5}{n(\ln(n))^2} = 5\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$

We compute $5\int_2^{\infty}\frac{1}{x(\ln(x))^2}\,dx$ let $u=\ln(x)$ $du=\frac{1}{x}dx$

$= 5\int_2^{\infty}\frac{1}{u^2}\,du = 5\left[-\frac{1}{u}\right]_2^{\infty} = 5\left[\frac{-1}{\ln(x)}\right]_2^{\infty}$

$= \underbrace{-5\lim_{x\to\infty}\frac{1}{\ln(x)}}_{\text{limit is } 0} + 5\frac{1}{\ln(2)} = \frac{5}{\ln(2)}$

so $\sum_{n=2}^{\infty}\frac{5}{n(\ln(n))^2}$ converges by the integral test

4) $\sum_{n=2}^{\infty} \frac{n}{n^4-1}$ ← ill-posed for $n=1$

$u = x^2 \quad du = 2x\,dx$

We compute $\int_2^{\infty} \frac{x}{x^4-1}\,dx = \frac{1}{2}\int_2^{\infty} \frac{1}{u^2-1}\,du = \frac{1}{2}\int_2^{\infty} \frac{1}{(u-1)(u+1)}\,du$

We can write $\frac{1}{(u-1)(u+1)} = \frac{A}{(u-1)} + \frac{B}{(u+1)}$, so $A(u+1)+B(u-1) = 1$

or $(A+B)u + (A-B)1 = 1$

Here $A = \frac{1}{2}$, $B = \frac{-1}{2}$, so

$$\frac{1}{2}\int_2^{\infty} \frac{1}{(u-1)(u+1)}\,du = \frac{1}{2}\int_2^{\infty} \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \frac{1}{4}\int_2^{\infty} \frac{1}{u-1}\,du - \frac{1}{4}\int_2^{\infty} \frac{1}{u+1}\,du$$

$$= \frac{1}{4}\Big[\ln(u-1) - \ln(u+1)\Big]_2^{\infty} = \frac{1}{4}\Big[\ln\left(\frac{u-1}{u+1}\right)\Big]_2^{\infty}$$

$$= \frac{1}{4}\Big[\ln\left(\frac{x^2-1}{x^2+1}\right)\Big]_2^{\infty} = \frac{1}{4}\lim_{x\to\infty} \ln\left(\frac{x^2-1}{x^2+1}\right) - \frac{1}{4}\ln\left(\frac{3}{5}\right)$$

$$= \frac{1}{4}\ln\left(\lim_{x\to\infty} \frac{x^2-1}{x^2+1}\right) - \frac{1}{4}\ln\left(\frac{3}{5}\right) = \frac{1}{4}\ln(1) - \frac{1}{4}\ln\left(\frac{3}{5}\right) = \frac{1}{4}\ln\left(\frac{3}{5}\right)$$

so $\sum_{n=2}^{\infty} \frac{n}{n^4-1}$ converges by the integral test.

★ Alternatively, $\frac{n}{n^4-1}$ is a decreasing sequence

and $\frac{n}{n^4-1} \leq \frac{2}{n^3}$, so $\sum_{n=2}^{\infty} \frac{n}{n^4-1}$ converges

by the comparison test, since $\sum_{n=2}^{\infty} \frac{2}{n^3}$ is convergent.

5) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ Note $0 \leq \cos^2 n \leq 1$, so $\frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series,

by the comparison test we see $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ converges.

6) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$

$u = \ln(x)$ $\quad v = -\frac{1}{2}x^{-2}$

$du = \frac{1}{x}dx$ $\quad dv = x^{-3}dx$

We compute $\int_1^{\infty} \underbrace{\frac{\ln(x)}{x^3}dx}_{u\,dv} = \left[\frac{1}{2}\underbrace{\ln(x)}_{u}\underbrace{x^{-2}}_{v} - \int \underbrace{-\frac{1}{2}x^{-2}}_{v} \cdot \underbrace{\frac{1}{x}dx}_{du}\right]_1^{\infty}$

$$= \left[-\frac{1}{2}\ln(x)/x^2 - \frac{1}{4x^2}\right]_1^{\infty}$$

$$= \lim_{x\to\infty}\left(-\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4x^2}\right) + \frac{1}{2}\frac{\ln(1)}{1} + \frac{1}{4}$$

$$= \frac{1}{4}$$

so $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ converges by the integral test.

Alternatively, $0 \le \frac{\ln(n)}{n^3} \le \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent,

so $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges by the comparison test.

7) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n^2}$

$u = \frac{1}{x}$ $\quad du = -\frac{1}{x^2}dx$

We compute $\int_1^{\infty} \frac{\sin(\frac{1}{x})}{x^2}dx = -\int_1^{\infty} \sin(u)\,du$

$$= \left[\cos(u)\right]_1^{\infty} = \left[\cos\left(\tfrac{1}{x}\right)\right]_1^{\infty} = \lim_{x\to\infty}\cos\left(\tfrac{1}{x}\right) - \cos(1)$$

$$= \cos\left(\lim_{x\to\infty}\tfrac{1}{x}\right) - \cos(1) = \cos(0) - \cos(1)$$

Thus by the integral test, $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$ converges.

Alternatively, $0 \le \sin(1/n) \le 1$ (Why?), so $0 \le \frac{\sin(1/n)}{n^2} \le \frac{1}{n^2}$,

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$ converges by the comparison test.

8) What is the smallest k for which the integral approximation guarantees

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{k} \frac{1}{n^2} \leq \frac{1}{1000} ?$$

We compute

$$\int_{k}^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{k}^{\infty} = \lim_{x\to\infty} -\frac{1}{x} + \frac{1}{k} = \frac{1}{k}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{k} \frac{1}{n^2} = \sum_{n=k+1}^{\infty} \frac{1}{n^2} \leq \int_{k}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{k}$,

we must set $k = 1000$ to guarantee this accuracy.

Note $\displaystyle\sum_{n=1}^{1000} \frac{1}{n^2} = 1.6439345\ldots$

while $\displaystyle\frac{\pi^2}{6} = 1.6449340\ldots$ Interesting!